Classical differential geometry of two-dimensional surfaces

1 Basic definitions

This section gives an overview of the basic notions of differential geometry for two-dimensional surfaces. It follows mainly Kreyszig [Kre91] in its discussion.

Definition of a surface

Let us consider the vector function $\boldsymbol{X}(\xi^1, \xi^2) \in \mathbb{R}^3$ with

$$\boldsymbol{X} : \mathbb{R}^2 \supset \Xi \ni (\xi^1, \xi^2) \mapsto \boldsymbol{X}(\xi^1, \xi^2) \in U \subset \mathbb{R}^3 \ , \tag{1}$$

where Ξ is an open subset of $\mathbb{R}^2$. Let $\boldsymbol{X}(\xi^1, \xi^2)$ be of class $r \geq 1$ in Ξ, which means that one of its component functions X_i $(i \in \{x, y, z\})$ is of class r and the other ones are at least of this class.[1] Let furthermore the Jacobian matrix $\frac{\partial(X_x, X_y, X_z)}{\partial(\xi^1, \xi^2)}$ be of rank 2 in Ξ which implies that the vectors

$$\boldsymbol{e}_a := \frac{\partial \boldsymbol{X}}{\partial \xi^a} = \partial_a \boldsymbol{X} \ , \quad a \in \{1, 2\} \ , \tag{2}$$

are linearly independent. The mapping (1) then defines a smooth two-dimensional surface patch U embedded in three-dimensional Euclidean space $\mathbb{R}^3$ with coordinates ξ^1 and ξ^2 (see Fig. 1). A union Σ of surface patches is called a surface if two arbitrary patches U and U' of Σ can be joined by finitely many patches $U = U_1, U_2, \ldots, U_{n-1}, U_n = U'$ in such a way that the intersection of two subsequent patches is again a surface patch [Kre91, p. 76]. To simplify the following let us restrict ourselves to a surface that can be covered by one patch U only.
The vectors $\boldsymbol{e}_a$, defined in Eqn. (2), are the tangent vectors of the surface. They are not normalized in general. Together with the unit normal

$$\boldsymbol{n} := \frac{\boldsymbol{e}_1 \times \boldsymbol{e}_2}{|\boldsymbol{e}_1 \times \boldsymbol{e}_2|} \ , \tag{3}$$

they form a local basis (*local frame*) in $\mathbb{R}^3$ (see Fig. 2):

$$\boldsymbol{e}_a \cdot \boldsymbol{n} = 0 \ , \quad \text{and} \quad \boldsymbol{n} \cdot \boldsymbol{n} = 1. \tag{4}$$

[1] A function of one or several variables is called a *function of class r* if it possesses continous partial derivatives up to order r.

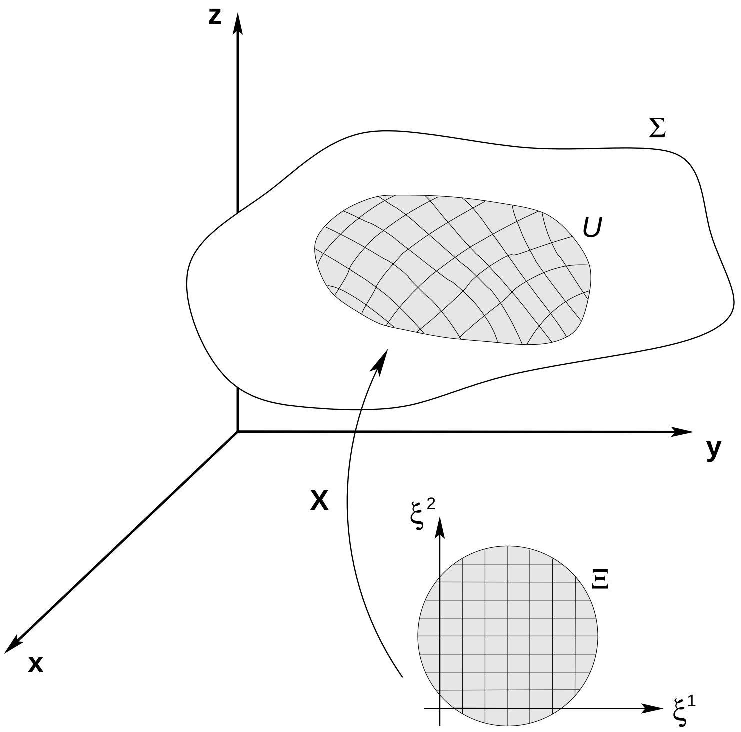

Figure 1: Parametrization of a surface

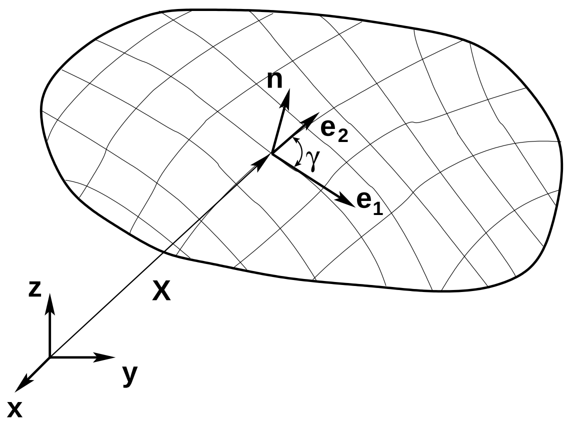

Figure 2: Local frame on the surface

The metric tensor (first fundamental form)

With the tangent vectors $\boldsymbol{e}_a$, one can define the *metric tensor* (also called the *first fundamental form*)

$$g_{ab} := \boldsymbol{e}_a \cdot \boldsymbol{e}_b \ . \tag{5}$$

This covariant second rank tensor is symmetric ($g_{ab} = g_{ba}$) and positive definite [Kre91, p. 86]. It helps to determine the infinitesimal Euclidean distance in terms of the coordinate differentials [Kre91, p. 82]

$$\begin{aligned} \mathrm{d}s^2 &= [\boldsymbol{X}(\xi^1 + \mathrm{d}\xi^1, \xi^2 + \mathrm{d}\xi^2) - \boldsymbol{X}(\xi^1, \xi^2)]^2 = (\boldsymbol{e}_1\, \mathrm{d}\xi^1 + \boldsymbol{e}_2\, \mathrm{d}\xi^2)^2 \\ &= (\boldsymbol{e}_a\, \mathrm{d}\xi^a)^2 = (\boldsymbol{e}_a \cdot \boldsymbol{e}_b)\, \mathrm{d}\xi^a\, \mathrm{d}\xi^b \\ &= g_{ab}\, \mathrm{d}\xi^a\, \mathrm{d}\xi^b \ , \end{aligned} \tag{6}$$

where the sum convention is used in the last two lines (see App. **??**). The contravariant dual tensor of the metric may be defined via

$$g_{ac}\, g^{cb} := \delta_a^b := \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases} , \tag{7}$$

where δ_a^b is the Kronecker symbol. The metric and its inverse can be used to raise and lower indices in tensor equations. Consider for instance the second rank tensor t_{ab}:

$$\text{Raising:} \qquad t_{ac}\, g^{cb} = t_a{}^b \ , \qquad\qquad \text{and lowering:} \qquad t_a{}^c\, g_{cb} = t_{ab} \ . \tag{8}$$

The determinant of the metric[2]

$$g := \det \mathbf{g} = |g_{ab}| = g_{11} g_{22} - g_{12} g_{21} \tag{9}$$

can be exploited to calculate the infinitesimal area element $\mathrm{d}A$: let γ be the angle between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ (see Fig. 2). Then

$$\begin{aligned} |\boldsymbol{e}_1 \times \boldsymbol{e}_2|^2 &= |\boldsymbol{e}_1|^2 |\boldsymbol{e}_2|^2 \sin^2\gamma = g_{11} g_{22}(1 - \cos^2\gamma) = g_{11} g_{22} - (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)^2 \\ &= g_{11} g_{22} - g_{12} g_{12} = g \ , \end{aligned} \tag{10}$$

and thus

$$\mathrm{d}A = |\boldsymbol{e}_1 \times \boldsymbol{e}_2|\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2 = \sqrt{g}\, \mathrm{d}^2\xi \ . \tag{11}$$

The covariant derivative

The partial derivative ∂_a is itself not a tensor. One therefore defines the covariant derivative ∇_a on a tensor $t^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m}$

$$\begin{aligned} \nabla_c t^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m} &= \partial_c t^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m} \\ &\quad + t^{d\, a_2 \ldots a_n}_{b_1 b_2 \ldots b_m} \Gamma_{dc}^{\ \ a_1} + t^{a_1 d \ldots a_n}_{b_1 b_2 \ldots b_m} \Gamma_{dc}^{\ \ a_2} + \ldots + t^{a_1 a_2 \ldots d}_{b_1 b_2 \ldots b_m} \Gamma_{dc}^{\ \ a_n} \\ &\quad - t^{a_1 a_2 \ldots a_n}_{d\, b_2 \ldots b_m} \Gamma_{b_1 c}^{\ \ d} - t^{a_1 a_2 \ldots a_n}_{b_1 d \ldots b_m} \Gamma_{b_2 c}^{\ \ d} - \ldots - t^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots d} \Gamma_{b_m c}^{\ \ d} \ , \end{aligned} \tag{12}$$

[2] Note that $\mathbf{g}$ is the matrix consisting of the metric tensor components g_{ab}.

where the ${\Gamma_{ab}}^c$ are the *Christoffel symbols of the second kind* with

$$ {\Gamma_{ab}}^c = (\partial_a \boldsymbol{e}_b) \cdot \boldsymbol{e}^c \ , \tag{13} $$

and ∇_a is now a tensor. For the covariant differentiation of sums and products of tensors the usual rules of differential calculus hold. The metric-compatible Laplacian Δ can be defined as $\Delta := \nabla_a \nabla^a$.
Note in particular that

$$ \nabla_a \boldsymbol{e}_b = \partial_a \boldsymbol{e}_b - {\Gamma_{ab}}^c \boldsymbol{e}_c \ , \qquad \text{and} \tag{14} $$

$$ \nabla_a g_{bc} = \nabla_a g^{bc} = \nabla_a g = 0 \ . \tag{15} $$

Equation (15) is also called the *Lemma of Ricci*. It implies that raising and lowering of indices commutes with the process of covariant differentiation.

Orientable surfaces

The orientation of the normal vector $\boldsymbol{n}$ in one point S of the surface depends on the choice of the coordinate system [Kre91, p. 108]: exchanging, for instance, ξ^1 and ξ^2 also flips $\boldsymbol{n}$ by 180 degrees. A surface is called *orientable* if no closed curve $\mathcal{C}$ through any point S of the surface exists which causes the sense of $\boldsymbol{n}$ to change when displacing $\boldsymbol{n}$ continuously from S along $\mathcal{C}$ back to S. An example of a surface that is not orientable is the *Möbius strip*.

The extrinsic curvature tensor (second fundamental form)

Two surfaces may have the same metric tensor g_{ab} but different curvature properties in $\mathbb{R}^3$. In order to describe such properties let us consider a surface Σ of class[3] $r \geq 2$ and a curve $\mathcal{C}$ of the same class on Σ with the parametrization $\boldsymbol{X}(\xi^1(s), \xi^2(s))$ on Σ, where s is the arc length of the curve (see Fig. 3).
At every point of the curve where its curvature $k > 0$, one may define a moving trihedron $\{\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{b}\}$ where $\boldsymbol{t} = \dot{\boldsymbol{X}}$ is the unit tangent vector, $\boldsymbol{p} = \dot{\boldsymbol{t}}/|\dot{\boldsymbol{t}}| = \dot{\boldsymbol{t}}/k$ is the unit principal normal vector, and $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{p}$ is the unit binormal vector of the curve.[4]
Furthermore, let η be the angle between the unit normal vector $\boldsymbol{n}$ of the surface and the unit principal normal vector $\boldsymbol{p}$ of the curve with $\cos\eta = \boldsymbol{p} \cdot \boldsymbol{n}$ (see again Fig. 3).
The curvature k of the curve can then be decomposed into a part which is due to the fact that the *surface* is curved in $\mathbb{R}^3$ and a part due to the fact that the *curve* itself is curved. The former will be called the normal curvature $K_{\rm n}$, the latter the geodesic curvature $K_{\rm g}$. One defines:

$$ K_{\rm n} := -\dot{\boldsymbol{t}} \cdot \boldsymbol{n} = -k\ (\boldsymbol{p} \cdot \boldsymbol{n}) = -k \cos\eta \ , \qquad \text{and} \tag{16} $$

$$ K_{\rm g} := \boldsymbol{t} \cdot (\dot{\boldsymbol{t}} \times \boldsymbol{n}) = k\, \boldsymbol{t} \cdot (\boldsymbol{p} \times \boldsymbol{n}) = k \sin\eta\ \mathrm{sign}\,(\boldsymbol{n} \cdot \boldsymbol{b}) \ . \tag{17} $$

[3] This means that its parametrization $\boldsymbol{X}(\xi^1, \xi^2)$ is of class $r \geq 2$.
[4] The dot denotes the derivative with respect to the arc length s.

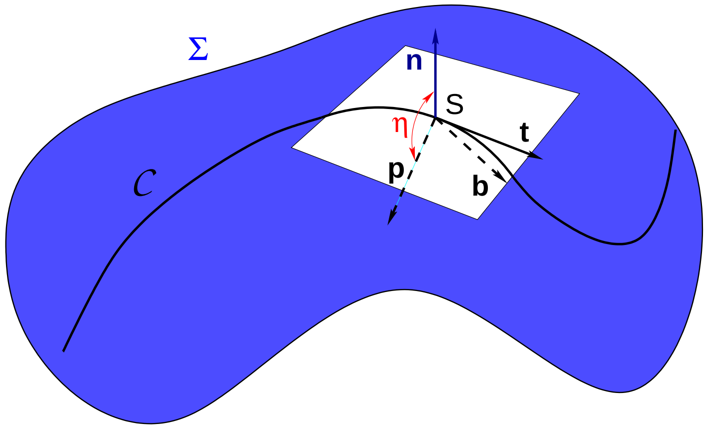

Figure 3: Curve on a surface

Here, we are interested in the curvature properties of the surface. Therefore, the normal curvature K_{n} is the relevant quantity that has to be studied a bit further.[5] The vector $\dot{\boldsymbol{t}}$ may be written as

$$\dot{\boldsymbol{t}} = \ddot{\boldsymbol{X}} = \frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{e}_a \dot{\xi}^a) = (\partial_b \boldsymbol{e}_a)\, \dot{\xi}^a \dot{\xi}^b + \boldsymbol{e}_a \ddot{\xi}^a \ . \tag{18}$$

Thus, Eqn. (16) turns into

$$K_{\mathrm{n}} = -k \cos\eta = (-\boldsymbol{n} \cdot \partial_a \boldsymbol{e}_b)\, \dot{\xi}^a \dot{\xi}^b \ , \tag{19}$$

where it has been exploited that $\partial_a \boldsymbol{e}_b = \partial_b \boldsymbol{e}_a$. The expression in brackets is the *extrinsic curvature tensor* or *second fundamental form*

$$K_{ab} := -\boldsymbol{n} \cdot \partial_a \boldsymbol{e}_b = \boldsymbol{e}_a \cdot \partial_b \boldsymbol{n} \ . \tag{20}$$

It is a symmetric covariant second rank tensor such as the metric. The second relation in Eqn. (20) follows if one differentiates the first equation of (4) with respect to ξ^a.

The extrinsic curvature can be written covariantly:

$$K_{ab} := -\boldsymbol{n} \cdot \nabla_a \boldsymbol{e}_b \ . \tag{21}$$

This is possible because $\partial_a \boldsymbol{e}_b$ differs from $\nabla_a \boldsymbol{e}_b$ only by terms proportional to the tangent vectors $\boldsymbol{e}_c$, which vanish when multiplied by $\boldsymbol{n}$ (see Eqn. (14)).

[5] The minus sign in the definition of K_{n}, Eqn. (16), is unfortunately a matter of convention and is here chosen in accordance to the literature where the surface stress tensor for fluid membranes has been introduced [CG02, Guv04]. A sphere with outward pointing unit normal has a positive normal curvature then. Note that this differs from Ref. [Kre91].

One can easily see from Eqn. (20) that K_{ab} has got something to do with curvature: at every point of the surface it measures the change of the normal vector in $\mathbb{R}^3$ for an infinitesimal displacement in the direction of a coordinate curve.
To learn more about the normal curvature let us consider a reparametrization of the curve $\mathcal{C}$ with the new parameter t. One gets

$$\dot{\xi}^a = \frac{\mathrm{d}\xi^a}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\xi^{a\prime}}{s'} \ , \tag{22}$$

where $'$ denotes the derivative with respect to t. Equation (19) thus takes the form

$$K_\mathrm{n} = K_{ab}\,\dot{\xi}^a\dot{\xi}^b = \frac{K_{ab}\,\xi^{a\prime}\xi^{b\prime}}{(s')^2} \overset{(6)}{=} \frac{K_{ab}\,\xi^{a\prime}\xi^{b\prime}}{g_{ab}\,\xi^{a\prime}\xi^{b\prime}} = \frac{K_{ab}\,\mathrm{d}\xi^a\,\mathrm{d}\xi^b}{g_{ab}\,\mathrm{d}\xi^a\,\mathrm{d}\xi^b} \ . \tag{23}$$

For a fixed point S, K_{ab} and g_{ab} are fixed as well. The value of K_n then only depends on the direction of the tangent vector $\boldsymbol{t}$ of the curve. One may search for extremal values of K_n at S by rewriting Eqn. (23):

$$(K_{ab} - K_\mathrm{n} g_{ab})\,\dot{\xi}^a\dot{\xi}^b = 0 \ . \tag{24}$$

A differentiation with respect to $\dot{\xi}^c$ yields the result

$$(K_{ac} - K_\mathrm{n} g_{ac})\,\dot{\xi}^a = 0 \ , \tag{25}$$

because $\mathrm{d}K_\mathrm{n} = 0$ is necessary for K_n to be extremal. Through the raising of one index, Eqn. (25) becomes an eigenvalue problem for K_a^b. Its eigenvectors are the tangent directions along which the normal curvature is extremal. They are called *principal directions* and are orthogonal to each other [Kre91, p. 129]. The eigenvalues will be called the *principal curvatures* k_1 and k_2 of the surface in point S. All other values of K_n in S in any direction can be calculated via Euler's theorem [Kre91, p. 132]. If the curve follows a principal direction at every point, it is also called a *line of curvature*.
For an arbitrary curve on the surface the symbol $K_\parallel$ denotes the normal curvature belonging to the direction the curve is following, whereas $K_\perp$ denotes the normal curvature belonging to the direction perpendicular to the curve in every point.
It is useful to define the following two notions: the *total curvature*

$$K := g^{ab}K_{ab} = K_a^a = k_1 + k_2 \ , \tag{26}$$

and the *Gaussian curvature*

$$K_\mathrm{G} := |K_a^b| = k_1 k_2 \ . \tag{27}$$

The quantities $|K|$ and K_G are invariant under surface reparametrizations because they only involve the eigenvalues of the extrinsic curvature tensor. They occur, for instance, in the surface Hamiltonian of a fluid membrane. Note that one can rewrite K_G

$$K_\mathrm{G} = |K_a^b| = |K_{ac}g^{cb}| = |K_{ac}|\,|g^{cb}| = \frac{K_{11}K_{22} - K_{12}K_{21}}{g} \ . \tag{28}$$

The equations of Gauss and Weingarten

With the help of the extrinsic curvature it is also possible to find relations for the partial derivatives of the local frame vectors: the normal vector $\boldsymbol{n}$ is a unit vector (see Eqn. (4)) and therefore

$$\boldsymbol{n} \cdot \partial_a \boldsymbol{n} = 0 \ . \tag{29}$$

Thus, $\partial_a \boldsymbol{n}$ is a linear combination of the tangent vectors $\boldsymbol{e}_a$. We know that $\partial_a \boldsymbol{n} \cdot \boldsymbol{e}_a = K_{ab}$ (see Eqn. (20)), which yields the *Weingarten equations*

$$\partial_a \boldsymbol{n} = \nabla_a \boldsymbol{n} = K_a^b \boldsymbol{e}_b \ . \tag{30}$$

For the tangent vector $\boldsymbol{e}_a$ a decomposition yields

$$\partial_a \boldsymbol{e}_b = (\boldsymbol{n} \cdot \partial_a \boldsymbol{e}_b)\boldsymbol{n} + (\boldsymbol{e}^c \cdot \partial_a \boldsymbol{e}_b)\boldsymbol{e}_c \overset{(20),(13)}{=} -K_{ab}\boldsymbol{n} + \Gamma_{ab}{}^c \boldsymbol{e}_c \ . \tag{31}$$

These are the *Gauss equations*, which can be rewritten covariantly:

$$\nabla_a \boldsymbol{e}_b \overset{(14)}{=} -K_{ab}\boldsymbol{n} \ . \tag{32}$$

Intrinsic curvature and integrability conditions

Do the partial differential Eqns. (30) and (32) have solutions for any chosen g_{ab} and K_{ab}? The answer is no; certain integrability conditions have to be satisfied. We require the embedding functions $\boldsymbol{X}$ to be of class $r \geq 3$ and

$$\partial_a \partial_b \boldsymbol{e}_c = \partial_b \partial_a \boldsymbol{e}_c \ . \tag{33}$$

From this follows [Kre91, p. 142 et seq.]

$$R^a{}_{bcd} = K_{bd}K_c^a - K_{bc}K_d^a \ , \qquad \text{and} \tag{34}$$

$$\nabla_a K_{bc} = \nabla_b K_{ac} \ , \tag{35}$$

where

$$R^a{}_{bcd} := \partial_c \Gamma_{bd}{}^a - \partial_d \Gamma_{bc}{}^a + \Gamma_{bd}{}^e \Gamma_{ec}{}^a - \Gamma_{bc}{}^e \Gamma_{ed}{}^a \ , \tag{36}$$

is called the *mixed Riemann curvature tensor*. It is intrinsic because it does not depend on the normal vector $\boldsymbol{n}$. Expression (35) is also referred to as the *equation of Mainardi-Codazzi.*

The *Ricci tensor* is defined as the contraction of the Riemann tensor with respect to its first and third index:

$$R_{ab} := R^c{}_{acb} \ . \tag{37}$$

A further contraction of the Ricci tensor yields the intrinsic *scalar curvature* of the surface (Ricci scalar)

$$\mathcal{R} := g^{ab} R_{ab} \ . \tag{38}$$

From Eqn. (34) one then obtains

$$R_{ab} = K K_{ab} - K_{ac} K^c_b \ , \qquad \text{and} \tag{39}$$

$$\mathcal{R} = K^2 - K^{ab} K_{ab} \ . \tag{40}$$

Combining Eqn. (28) with the completely covariant form of Eqn. (34), one gets after a few calculations:

$$R_{ab} = K_{\rm G}\, g_{ab} \ , \qquad \text{and} \tag{41}$$

$$\mathcal{R} = 2\, K_{\rm G} \ . \tag{42}$$

These equations confirm Gauss' Theorema Egregium, which states that the Gaussian curvature, even though originally defined in an extrinsic way, in fact only depends on the first fundamental form [Kre91, p. 145] and is thus an intrinsic surface property.

2 Gauss-Bonnet theorem

The Gauss-Bonnet theorem for simply connected surfaces

The Gauss-Bonnet theorem states the following [Kre91, p. 169]: Let Σ_0 be a simply connected surface patch of class $r_{\Sigma_0} \geq 3$ with simple closed boundary $\partial\Sigma_0$ of class $r_{\partial\Sigma_0} \geq 3$. Furthermore, let $\boldsymbol{X}(\xi^1(s), \xi^2(s))$ be the parametrization of the boundary curve, where s is the arc length. Then

$$\int_{\partial\Sigma_0} {\rm d}s\ K_{\rm g} + \int_{\Sigma_0} {\rm d}A\ K_{\rm G} = 2\pi \ , \tag{43}$$

where ${\rm d}A$ is the infinitesimal area element, $K_{\rm g}$ is the geodesic curvature of $\partial\Sigma_0$, and $K_{\rm G}$ is the Gaussian curvature of Σ_0. Note that the integration along the boundary curve has to be carried out in such a sense that the right-hand rule is satisfied: take your right thumb and point it in the direction of the normal vector $\boldsymbol{n}$. If you then curl your fingers, the tips indicate the direction of integration.

One can check the consistency of Eqn. (43) easily by considering a flat circle with radius a: Its Gaussian curvature is zero and therefore also the integral over it. The geodesic curvature, however, is equal to $1/a$ in every point of the boundary. Thus, the integral over $K_{\rm g}$ yields $2\pi a \times 1/a$, which is equal to the right-hand side of Eqn. (43).

Generalization to multiply connected surfaces

A generalization of this theorem to multiply connected surfaces is also possible [Kre91, p. 172]: One can cut multiply connected surfaces into simply connected ones. Take, for instance, a surface as in Fig. 4. The path of integration along the

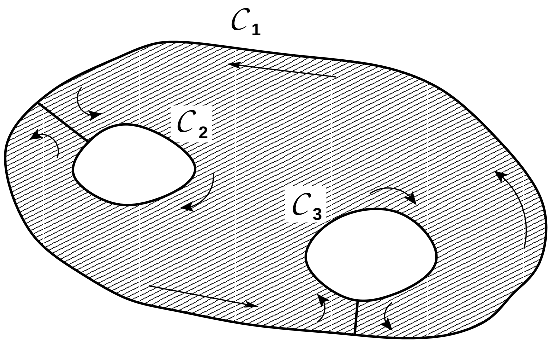

Figure 4: Integration contour for multiply connected surface patches

boundary may be chosen as depicted by the arrows. The sections are passed twice in opposite directions; their contributions therefore cancel each other. The end points of every section, however, add a term of π each to the integral $\int \mathrm{d}s\, K_{\mathrm{g}}$. This is due to the rotation the tangent makes at each of these points. Every section therefore contributes 2π to the integral. For the case of Fig. 4 we thus have an extra term of 4π.

Application to closed surfaces

It is also possible to apply the Gauss-Bonnet theorem to closed surfaces [Kre91, p. 172]. Topologically, any closed orientable surface is homeomorphic[6] to a sphere with p attached "handles". This number p is also called *genus* of the surface. Consequently, a sphere has genus 0, a torus genus 1, etc. One then obtains for any closed orientable surface Σ of genus p [Kre91, p. 172]:

$$\int_\Sigma \mathrm{d}A\, K_{\mathrm{G}} = 4\pi(1-p) \; . \tag{44}$$

This implies that the integral over the Gaussian curvature is a topological invariant for any closed surface with fixed genus p.

3 Monge parametrization

For surfaces with no "overhangs", it is sufficient to describe their position in terms of a height $h(x, y)$ above the underlying reference plane as a function of the orthonormal coordinates x and y. The direction of the basis vectors $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\} \in \mathbb{R}^3$ is chosen as depicted in Fig. 5.

[6] This means that the mapping and its inverse are continuous and bijective.

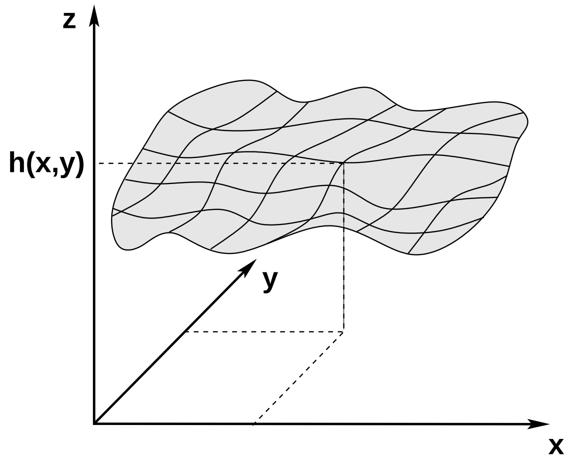

Figure 5: Monge parametrization

The tangent vectors on the surface can then be expressed as $\boldsymbol{e}_x = (1, 0, h_x)^{\mathrm{T}}$ and $\boldsymbol{e}_y = (0, 1, h_y)^{\mathrm{T}}$, where $h_i = \partial_i h$ $(i, j \in \{x, y\})$. The metric is equal to

$$g_{ij} = \delta_{ij} + h_i h_j \ , \tag{45}$$

where δ_{ij} is the Kronecker symbol. We also define $\boldsymbol{\nabla} = (\partial_x, \partial_y)^{\mathrm{T}}$. The metric determinant and the infinitesimal surface element can then be written as

$$g = |g_{ij}| = 1 + (\boldsymbol{\nabla} h)^2 \quad \text{and} \tag{46}$$

$$\mathrm{d}A = \sqrt{g}\, \mathrm{d}x\, \mathrm{d}y \ . \tag{47}$$

The inverse metric is given by

$$g^{ij} = \delta_{ij} - \frac{h_i h_j}{g} \ . \tag{48}$$

Note that Eqns. (45) and (48) are not tensor equations. The right-hand side gives merely numerical values for the components of the covariant tensors g_{ij} and g^{ij}. The unit normal vector is equal to

$$\boldsymbol{n} = \frac{1}{\sqrt{g}} \begin{pmatrix} -\boldsymbol{\nabla} h \\ 1 \end{pmatrix} \ . \tag{49}$$

With the help of Eqn. (20) the extrinsic curvature tensor can be calculated:

$$K_{ij} = -\frac{h_{ij}}{\sqrt{g}} \ , \tag{50}$$

where $h_{ij} = \partial_i \partial_j h$. Note that Eqn. (50) again is not a tensor equation and gives only numerical values for the components of K_{ij}.

Finally, it is also possible to write the total curvature K in Monge parametrization:

$$K = -\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\nabla} h}{\sqrt{g}} \right) . \tag{51}$$

One is often interested in surfaces that deviate only weakly from a flat plane. In this situation the gradients h_i are small. Therefore, it is enough to consider only the lowest nontrivial order of a small gradient expansion. K and $\mathrm{d}A$ can then be written as

$$K = -\boldsymbol{\nabla}^2 h + \mathcal{O}[(\boldsymbol{\nabla} h)^2] , \tag{52}$$

$$\mathrm{d}A = \left\{ 1 + \frac{1}{2}(\boldsymbol{\nabla} h)^2 + \mathcal{O}[(\boldsymbol{\nabla} h)^4] \right\} \mathrm{d}x \, \mathrm{d}y . \tag{53}$$

References

[CG02] R. Capovilla and J. Guven. Stresses in lipid membranes. *J. Phys. A: Math. Gen.*, 35: pp. 6233–6247, 2002.

[Guv04] J. Guven. Membrane geometry with auxiliary variables and quadratic constraints. *J. Phys. A: Math. Gen.*, 37: L313–L319, 2004.

[Kre91] E. Kreyszig. *Differential geometry.* Dover, Mineola, NY, 1991.